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ON THE ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF BOUNDARY PROBLEMS FOR QUASILINEAR DIFFERENTIAL EQUATIONS /778**

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The construction of asymptotic solutions, involving the small parameter ϵ, of boundary problems for nonlinear and quasilinear differential equations is demonstrated on the example of ordinary differential equations in a direction transverse to the boundary. Under assumption of the existence of a bounded M-domain (which is defined) a theorem of asymptotic representation, valid for the solution $\tilde{y}_\varepsilon(x)$ of the ordinary differential equation lying in M, is proved by successive expansions in series of the linear case.

The method of construction of asymptotic solutions, involving the small parameter ϵ, of boundary problems for linear differential equations (Ref.1, 2) is also applicable to certain classes of nonlinear differential equations. We shall illustrate this on the example of the ordinary differential equations

$$L_\varepsilon y \equiv \varepsilon y'' + \varphi(x, y)y' - \psi(x, y) = 0, \quad y(0) = A, \quad y(1) = B. \tag{1}$$

The asymptotic solution of this problem in powers of the parameter A has been studied by Wasow (Ref.3). Consider the boundary equation

$$L_0 w \equiv \varphi(x, w)w' - \psi(x, w) = 0. \tag{2}$$

Let some domain D be covered by the bounding curves $w = w(x)$ [i.e., by solutions of eq.(2)]. Then, in the domain D, the values $w'(x) = \psi(x, y)/\varphi(x, y) = p(x, y)$ $(y = w(x))$ and $w''(x) = p'_x + p'_y p = q(x, y)$ are functions of (x, y).

Consider the case when, in the domain D,

$$\varphi(x, y) \geqslant \gamma > 0, \tag{3}$$

which, for the solutions (1), ensures the appearance of a boundary layer in the neighborhhod of $x = 0$. Let us apply the term "secant curve from above" (or "from below") for the solutions $\tilde{y}_\varepsilon(x)$ of eq.(1) to the curve $y = u(x)$ such that for $\tilde{y}_\varepsilon(x) \le u(x)$ $(\tilde{y}_\varepsilon(x) \ge u(x))$ the line $y = \tilde{y}_\varepsilon(x)$ cannot make contact within the zone $0 < x < 1$ with $y = u(x)$. For this it is sufficient that

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** Numbers given in the margin indicate pagination in the original foreign text.

$$L_\varepsilon u \equiv \varepsilon u'' + \varphi(x, u)[u' - p(x, u)] < 0 \quad (> 0)$$

for $0 < x < 1$. [For example, a secant from above will be the segment y = const on which $\psi > 0$, or the bounding curve $y = w(x)$ on which $q < 0$.]

Let there exist in the zone $0 < x < 1$ a domain satisfying the following conditions (let us call this the M-domain): 1) The domain is covered by a field of bounding curves $y = w(x)$, joining the points of the straight lines $x = 0$, $x = 1$; 2) the domain is bounded by the segments $[A_0, A_1]$ and $[B_0, B_1]$ of the straight lines $x = 0$ and $x = 1$ and by the curves r_1 and r_2 secant from above and from below; 3) the segment $[B_0, B_1]$ of the straight line $x = 1$ contains a segment $[\overline{B}_0, \overline{B}_1]$ such that every bounding curve $y = w(x)$ originating in the point $[1, B]$, $\overline{B}_0 \leq B \leq \overline{B}_1$ will pass entirely within this M-domain if $0 \leq x \leq 1$. It is easy to convince oneself that the problem (1) at $A_0 < A < A_1$, $\overline{B}_0 < B < \overline{B}_1$, for sufficiently small $\epsilon > 0$, has a solution $y = \tilde{y}_\varepsilon(x)$ passing within the M-domain.

Unless otherwise specifically noted, we will assume below that the M-domain exists, and that for the initial values A and B the above inequalities are satisfied, and that eq.(3) is also satisfied. We note that every solution of eq.(1) lying in the M-domain is bounded: $|\tilde{y}_\varepsilon(x)| \leq C$; hence it is easy to derive $|\tilde{y}'_\varepsilon(x)| \leq C_1/\epsilon$. In investigating the solution $y = \tilde{y}_\varepsilon(x)$ it is convenient to use the function $z(x) = \tilde{y}'_\varepsilon(x) - p(x, \tilde{y}_\varepsilon(x))$. This satisfies the equation /779

$$\varepsilon z' = -\varphi_1(x, \tilde{y}_\varepsilon) z - \varepsilon q(x, \tilde{y}_\varepsilon), \quad \varphi_1 = \varphi + \varepsilon p'_y. \tag{4}$$

Obviously, for sufficiently small ϵ, we have $\varphi_1 \geq \gamma_1 > 0$. Thus, by solving the Cauchy problem for eq.(4) for the initial conditions at $x = 0$, it can be demonstrated that $z(x)$ is the sum of an exponentially decreasing term of the boundary-layer type and a term of the order ϵ. Hence it follows (for sufficiently small ϵ) that the solution $\tilde{y}_\varepsilon(x)$ of the problem (1) for $x \geq x_0$, where $x_0 = O(\epsilon|\ln \epsilon|)$, falls in the ϵ-neighborhood of the bounding curve $y = w(x)$ $(w(1) = B)$. For $0 < x < x_0$, the difference $v(x) = \tilde{y}_\varepsilon(x) - w(x)$ is a function of the boundary-layer type, where $v(x_0) = O(\epsilon)$, $v'(x_0) = O(\epsilon)$. For $0 < x < x_0$, we have $v'(x) = O(1/\epsilon)$, $\epsilon v' = O(1)$. Neglecting quantities of the order of $O(1)$, we can write, for the principal part v_0 of this difference v, the following equation:

$$\varepsilon v_0'' + \varphi(v_0 + a) v_0' = 0 \quad (v_0(0) = A - a,\ a = w(0);\ \varphi(y) = \varphi(0, y)).$$

This equation is easily solved in quadratures and, as can be verified, for $\varphi \geq \gamma > 0$, we have $v_0(x) = O(1) \exp(-\gamma x/\epsilon)$, $v'(x) = O(1/\epsilon) \exp(-\gamma x/\epsilon)$.

Theorem. If eq.(3) is satisfied in $\overline{M}$ and if $\varphi(x, y)$ and $\psi(x, y)$ are accordingly smooth, the following asymptotic representations will be valid for the solutions $\tilde{y}_\varepsilon(x)$ of problem (1) (where $A_0 < A < A_1$, $\overline{B}_0 < B < \overline{B}_1$), lying in M:

$$\tilde{y}_\varepsilon(x) = w_0(x) + v_0(x) + \tilde{R}_0(x), \quad \tilde{R}_0(x) = O(\varepsilon|\ln \varepsilon|), \tag{5}$$

$$\tilde{y}_\varepsilon(x) = \left[w_0(x) + \sum_{s=1}^{n} \varepsilon^s w_s(x)\right] + \left[v_0(x) + \sum_{s=1}^{n+1} \varepsilon^s v_s\right] + R_n(x),$$
$$R_n(x) = O(\varepsilon^{n+1}). \tag{6}$$

We present the scheme of proof for eq.(6). After separation of the principal terms $w_0(x) + v_0(x)$ of the asymptotic solution, we are able to linearize the equations determining the higher terms of this asymptotic solution. The construction of the asymptotic (6) is analogous to the process described elsewhere (Ref.1, 2) for the linear case. We stipulate that

$$L_\varepsilon \bar{w}_n = O(\varepsilon^{n+1}), \text{ where } \bar{w}_n = \sum_0^n \varepsilon^s w_s, \quad w_0(1) = \tilde{y}(1), w_s(1) = 0 \text{ at } s \geqslant 1. \tag{7}$$

Expanding the functions $\varphi(x, \bar{w}_n)$ and $\psi(x, \bar{w}_n)$ at the point $(x, w_0(x))$ in powers of ϵ and equating in eq.(7) all terms of the same power of ϵ, we obtain

$$\begin{gathered}\varphi(x, w_0) w_0' - \psi(x, w_0) = 0, \quad w_0(1) = B; \\ \varphi(x, w_0) w_k' + [\varphi_y'(x, w_0) w_0' - \psi_y'(x, w_0)] w_k = \Phi_k - w_{k-1}'', \quad w_k(0) = 0;\end{gathered} \tag{8}$$

where Φ_k is a function of $w_0, w_1, \ldots, w_{k-1}, w', \ldots, w_{k-1}'$. Thus the quantities w_k are successively determined by the aid of the solution of the linear equations (8); w_k, w_k', w_k'' are functions bounded on [0, 1]. To find the asymptotic solution of the boundary layer $\bar{v}_n = v_0 + \epsilon v_1 + \ldots + \epsilon^{n+1} v_{n+1}$ we start out from the equation

$$L_\varepsilon(\bar{v}_n + \bar{w}_n) - L_\varepsilon(\bar{w}_n) = O(\varepsilon^{n+1}), \quad (\bar{v}_n + \bar{w}_n)|_{x=0} = A, \tag{9}$$

from which, according to eq.(7), it follows that $L_\varepsilon(\bar{v}_n + \bar{w}_n) = O(\epsilon^{n+1})$. Let us introduce, as done before (Ref.1, 2), the variable $t = x/\epsilon$; in this variable

$$\varepsilon L_\varepsilon u \equiv u''(t) + \varphi(\varepsilon t, u) u_t' - \varepsilon \psi(\varepsilon t, u).$$

Let us expand the obtained function $\bar{w}_n = \sum^n \epsilon^s w_s$ in a series in powers of $x = \epsilon t$; remembering that $w(0) = a$, and grouping the terms by powers of ϵ, we obtain /780

$$\bar{w}_n(x) = \bar{w}_n(\varepsilon t) = a + \sum^n \varepsilon^s p_s(t) + O(\varepsilon^{n+1}), \tag{10}$$

where the $p_s(t)$ are polynomials in t. Substituting this expression for $\bar{w}_n$ in eq.(9), and expanding the coefficients $\varphi(\epsilon t, \bar{w}_n)$ and $\psi(\epsilon t, \bar{w}_n)$ in powers of ϵ, we obtain successively

$$v_0''(t) + \varphi(a + v_0) v_0'(t) = 0, \quad v_0|_{t=0} = A - w_0(0) = A - a; \tag{11}$$

$$v_k''(t) + \varphi(a + v_0) v_k'(t) + \varphi_y'(a + v_0) v_0' v_k(t) = \Psi_k \ (k = 0, 1, \ldots, n+1);$$
$$v_k(0) = -w_k(0) \ \text{at} \ 1 \leqslant k \leqslant n; \quad v_{n+1}(0) = 0, \tag{12}$$

where Ψ_k is a function of v_0, v_1, ..., v_{k-1}, and also of $p_s(t)$, i.e., of functions already found. In this case, we shall seek v_k as functions of the boundary-layer type ($v|_\infty = 0$), which replaces the second boundary condition. It can be proved by induction that all functions v_k ($k = 0, 1, \ldots, n + 1$) are functions of the boundary-layer type.

To evaluate $R_n(x)$ in eq.(6) we note that, according to eqs.(7) and (9) and setting $\tilde{y}_\varepsilon = \tilde{y}$, $\tilde{y}_1 = \tilde{y} - R_n (= \bar{v}_n + \bar{w}_n)$, we have

$$L_\varepsilon \tilde{y} - L_\varepsilon \tilde{y}_1 = -L_\varepsilon \tilde{y}_1 = O(\varepsilon^{n+1}), \tag{13}$$

i.e.,

$$\varepsilon R_n'' + [\varphi(x, \tilde{y}) \tilde{y}' - \varphi(x, \tilde{y}_1) \tilde{y}_1'] - [\psi(x, \tilde{y}) - \psi(x, \tilde{y}_1)] = O(\varepsilon^{n+1}). \tag{14}$$

If $z_1 = \tilde{y}_1' - p(x, \tilde{y}_1)$, we obtain for z_1 an equation differing from eq.(4) by the introduction of $O(\varepsilon^{n+1})$ on the right-hand side. Setting $\delta z = z - z_1$, we have

$$R_n' = \tilde{y}' - \tilde{y}_1' = \bar{p}_y R_n + \delta z; \qquad \bar{p}_y = p_y(x, \tilde{y}_1 + \theta R_n), \quad 0 < \theta < 1. \tag{15}$$

Furthermore, solving eq.(4) and the corresponding equation for z_1 and noting that both $z(0)$ and $z_1(0)$ will be of the order of $1/\varepsilon$, we obtain

$$\delta z(x) = O(1/\varepsilon) \exp[-\gamma_1 x/\varepsilon] + O(\varepsilon) R_n(\theta x) + O(\varepsilon^{n+1}). \tag{16}$$

Considering eq.(15) as an equation linear in R_n, solving it under the condition $R_n(1) = 0$, and making use of eq.(16), we obtain

$$R_n(x) = O(1) \exp[-\gamma_1 x/\varepsilon] + O(\varepsilon) R_n(\xi) + O(\varepsilon^{n+1}), \tag{17}$$

where ξ is the point at which $|R_n(x)|$ reaches its maximum. If $\xi \geq \varepsilon^{1-k}$, $0 < k < 1$, it follows from eq.(17) that

$$R_n(\xi) = O(1) \exp(-\gamma_1 \varepsilon^{-k}) + O(\varepsilon^{n+1}). \tag{18}$$

Let $0 < \xi < \varepsilon^{1-k}$. Then, integrating eq.(14) between ε^{1-k} and ξ, we obtain, since $R_n'(\xi) = 0$,

$$-\varepsilon R_n'(\varepsilon^{1-k}) + \int_{\tilde{y}(\varepsilon^{1-k})}^{\tilde{y}(\xi)} \varphi(y)\,dy - \int_{\tilde{y}_1(\varepsilon^{1-k})}^{\tilde{y}_1(\xi)} \varphi(y)\,dy + \int_{\varepsilon^{1-k}}^{\xi} \Phi\,dx + O(\varepsilon^{n+1}) = 0, \tag{19}$$

where, as easy to demonstrate, $\Phi = O(1)R_n + O(x)R_n'$ and $\int_{\varepsilon^{1-k}}^{\xi} \Phi dx = O(\varepsilon^{1-k})|R_n(\xi)|$.

Further, $\varepsilon R_n'(\varepsilon^{1-k})$, according to eqs.(15) and (16), equals $O(\varepsilon)|R_n(\xi)| + O(1) \exp[-\gamma_1 \varepsilon^{-k}] + O(\varepsilon^{n+1})$. The difference of the integrals in eq.(19) reduces to the integral of $\varphi(y)$ over the interval $(\tilde{y}_1(\varepsilon^{1-k}), \tilde{y}(\varepsilon^{1-k}))$ of length $|R_n(\varepsilon^{1-k})| = O(1) \exp(-\gamma_1 \varepsilon^{-k}) + O(\varepsilon)|R_n(\xi)| + O(\varepsilon^{n+1})$ and the interval $(\tilde{y}_1(\xi), \tilde{y}(\xi))$ of length $|R_n(\xi)|$. We note that the integral over the second interval, which we shall denote by P, exceeds modulo $\gamma|R_n(\xi)|$. The remaining terms in eq.(19) yield an expression of the form: /781

$$O(1)\exp(-\gamma_1\varepsilon^{-k}) + O(\varepsilon^{1-k})|R_n(\xi)| + O(\varepsilon^{n+1}). \tag{20}$$

Hence, according to eq.(19) and making use of the inequality $|P| \geq \gamma|R_n(\xi)|$, $\gamma > 0$, we obtain

$$|R_n(\xi)| = O(\varepsilon^{n+1}) + O(1)\exp(-\gamma_1\varepsilon^{-k}). \tag{21}$$

Since the second term of eq.(21) is of higher order than the first, we thus obtain $|R_n(\xi)| = O(\varepsilon^{n+1})$. The theorem is proved. We can similarly prove:

If in the M-domain, the conditions of the theorem being satisfied, two solutions $\tilde{y}(x)$ and $\tilde{\tilde{y}}(x)$ of the problem (1) exist, then

$$\tilde{y}(x) - \tilde{\tilde{y}}(x) = O(\exp(-\gamma_1\varepsilon^{-k})),$$

where k is any fixed number between 0 and 1, i.e., there will always be uniqueness and accuracy to within a quantity of an exponential order of smallness relative to ε.

Sufficient conditions of uniqueness in the M-domain will be the simultaneous satisfaction of the inequalities:

$$\varphi > \gamma > 0, \quad p_y > 0, \quad (A - a)\varphi_y' \geqslant 0. \tag{22}$$

Note. The above constructions are also applicable to equations of more general form, for instance $\varepsilon y'' + f(x, y, y') = 0$, with restrictions corresponding to those indicated above. It should be mentioned that, in this case, the boundary layer may have a weaker character of variation.

Notes on quasilinear partial differential equations. As shown elsewhere (Ref.1, 2), for the linear case the construction of the boundary layer reduces to the solution of an ordinary differential equation in a direction transverse to the boundary. Constructions of the same type are also applicable to certain classes of quasilinear elliptic partial differential equations. For instance, for the equation $\varepsilon^2\Delta u - \psi(\rho, \varphi, u) = 0$ under the conditions $u|_{\rho=0} = f(\varphi)$ ($\rho = 0$ being the equation of the boundary Γ), $\psi(\rho, \varphi, 0) = 0$, $\psi_u' \geq \gamma^2 > 0$, the solution of the boundary equation (at $\varepsilon = 0$) will be $w \equiv 0$, while for the boundary layer in first approximation we obtain the ordinary equation $\epsilon^2 A(\varphi) \frac{\partial^2 v}{\partial \rho^2} - \psi \cdot$ $\cdot\ (0, \varphi, v) = 0$, $v|_{\rho=0} = f(\varphi)$, which is analogous to eq.(1). For the following approximations, linear equations are obtained and an expansion of the type of eq.(6) takes place. By the same method, an asymptotic representation of the form of eq.(6) can be obtained, for instance, for the quasilinear elliptic equations $L_\varepsilon u = h$ with a small parameter and higher derivatives, provided that 1) for any value of ϵ, however small, there also exists a unique smooth solution $u_\varepsilon(x, y)$ of the boundary problem for $L_\varepsilon u = h$ which continuously (uniformly in ϵ) depends on h [it is easy to indicate the classes of such equations on the basis of S.N.Bernshteyn's work (Ref.4)]; 2) the solution w of the boundary equation (at $\epsilon = 0$) is sufficiently smooth; 3) the construction of the boundary layer reduces, for instance, to an ordinary differential equation of the form of eq.(1).

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